

# Sharpened Error Bound for Random Sampling Based $\ell_2$ Regression

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**Abstract.** Given a data matrix  $\mathbf{X} \in \mathbb{R}^{n \times d}$  and a response vector  $\mathbf{y} \in \mathbb{R}^n$ , suppose  $n > d$ , it costs  $\mathcal{O}(nd^2)$  time and  $\mathcal{O}(nd)$  space to solve the least squares regression (LSR) problem. When  $n$  and  $d$  are both large, exactly solving the LSR problem is very expensive. When  $n \gg d$ , one feasible approach to accelerating LSR is to randomly embed  $y$  and all columns of  $\mathbf{X}$  into the subspace  $\mathbb{R}^c$  where  $c \ll n$ ; the induced LSR problem has the same number of columns but much fewer number of rows, and the induced problem can be solved in  $\mathcal{O}(cd^2)$  time and  $\mathcal{O}(cd)$  space.

The leverage scores based sampling is an effective subspace embedding method and can be applied to accelerate LSR. It was shown previously that  $c = \mathcal{O}(d\epsilon^{-2} \log d)$  is sufficient for achieving  $1 + \epsilon$  accuracy. In this paper we sharpen this error bound, showing that  $c = \mathcal{O}(d \log d + d\epsilon^{-1})$  is enough for  $1 + \epsilon$  accuracy.

## 1 Introduction

Given  $n$  data instances  $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ , each of dimension  $d$ , and  $n$  responses  $y_1, \dots, y_n$ , it is of interesting to find a model  $\beta \in \mathbb{R}^d$  such that  $\mathbf{y} = \mathbf{X}\beta$ . If  $n > d$ , there will not in general exist a solution to the linear system, so we instead seek to find a model  $\beta_{\text{LSR}}$  such that  $\mathbf{y} \approx \mathbf{X}\beta_{\text{LSR}}$ . This can be formulated as the least squares regression (LSR) problem:

$$\beta_{\text{LSR}} = \underset{\beta \in \mathbb{R}^d}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{X}\beta\|_2^2. \quad (1)$$

Suppose  $n \geq d$ , in general it takes  $\mathcal{O}(nd^2)$  time and  $\mathcal{O}(nd)$  space to compute  $\beta_{\text{LSR}}$  using the iterative numerical algorithms like QR decomposition or the conjugate gradient method.

LSR is perhaps one of the most widely used method in data processing, however, solving LSR for big data is very time and space expensive. In the big-data problems where  $n$  and  $d$  are both large, the  $\mathcal{O}(nd^2)$  time complexity and  $\mathcal{O}(nd)$  space complexity makes LSR prohibitive. So it is of great interest to find efficient solution to the LSR problem. Fortunately, when  $n \gg d$ , one can use a small portion of the  $n$  instances instead of using the full data to approximately compute  $\beta_{\text{LSR}}$ , and the computation cost can thereby be significantly reduced.

Random sampling based methods [5, 9, 10] and random projection based methods [2, 6] have been applied to make LSR more efficiently solved.

Formally speaking, let  $\mathbf{S} \in \mathbb{R}^{c \times n}$  be a random sampling/projection matrix, we solve the following problem instead of (1):

$$\tilde{\beta}_{\mathbf{S}} = \underset{\beta \in \mathbb{R}^d}{\operatorname{argmin}} \|\mathbf{S}\mathbf{y} - \mathbf{S}\mathbf{X}\beta\|_2^2. \quad (2)$$

This problem can be solved in only  $\mathcal{O}(cd^2)$  time and  $\mathcal{O}(cd)$  space. If the random sampling/projection matrix  $\mathbf{S}$  is constructed using some special techniques, then it is ensured theoretically that  $\tilde{\beta}_{\mathbf{S}} \approx \beta_{\text{lsr}}$  and that

$$\|\mathbf{y} - \mathbf{X}\tilde{\beta}_{\mathbf{S}}\|_2^2 \leq (1 + \epsilon) \|\mathbf{y} - \mathbf{X}\beta_{\text{lsr}}\|_2^2 \quad (3)$$

hold with high probability. There are two criteria to evaluate random sampling/projection techniques based LSR.

- **Running Time.** That is, the total time complexity in constructing  $\mathbf{S} \in \mathbb{R}^{c \times n}$  and computing  $\mathbf{S}\mathbf{X} \in \mathbb{R}^{c \times d}$ .
- **Dimension after Projection.** Given an error parameter  $\epsilon$ , if there exists a polynomial function  $C(d, \epsilon)$  such that if  $c > C(d, \epsilon)$ , the inequality (3) holds with high probability for all  $\mathbf{X} \in \mathbb{R}^{n \times d}$  and  $\mathbf{y} \in \mathbb{R}^n$ . Obviously  $C(d, \epsilon)$  is the smaller the better because the induced problem (2) can be solved in less time and space if  $c$  is small.

## 2 Contribution

The leverage scores based sampling is an important random sampling technique widely studied and empirically evaluated in the literature [2, 4, 5, 7, 9, 11, 10, 13, 14]. When applied to accelerate LSR, error analysis of the leverage scores based sampling is available in the literature. It was shown in [4] showed that by using the leverage scores based sampling with replacement, when

$$c > \mathcal{O}(d^2 \epsilon^{-2}),$$

the inequality (3) holds with high probability. Later on, [5] showed that by using the leverage scores based sampling without replacement,

$$c > \mathcal{O}(d \epsilon^{-2} \log d)$$

is sufficient to make (3) hold with high probability.

We provide a very simple proof to show that (3) holds with high probability when

$$c > \mathcal{O}(d \log d + d \epsilon^{-1}),$$

using the same leverage scores based sampling without replacement. Our results are described in Theorem 1. Our proof techniques are based on the previous work [5, 6], and our proof is self-contained.

**Algorithm 1** The Leverage Scores Based Sampling (without Replacement).

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- 1: **Input:** an  $n \times d$  real matrix  $\mathbf{X}$ , target dimension  $c < n$ .
  - 2: (Exactly or approximately) compute the leverage scores of  $\mathbf{X}$ :  $l_1, \dots, l_n$ ;
  - 3: Compute the sampling probabilities by  $p_i = \min\{1, cl_i/d\}$  for  $i = 1$  to  $n$ ;
  - 4: Denote the selected index set by  $\mathcal{C}$ , initialized by  $\emptyset$ ;
  - 5: For each index  $i \in [n]$ , add  $i$  to  $\mathcal{C}$  with probability  $p_i$ ;
  - 6: Compute the diagonal matrix  $\mathbf{D} = \text{diag}(p_1^{-1}, \dots, p_n^{-1})$ ;
  - 7: **return**  $\mathbf{S} \leftarrow$  the rows of  $\mathbf{D}$  indexed by  $\mathcal{C}$ .
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**Theorem 1.** Use the leverage score based sampling without replacement (Algorithm 1) to construct the  $c \times d$  sampling matrix  $\mathbf{S}$  where

$$c = \mathcal{O}(d \ln d + d\epsilon^{-1}),$$

and solve the approximate LSR problem (2) to obtain  $\tilde{\beta}_{\mathbf{S}}$ . Then with probability at least 0.8 the following inequalities hold:

$$\begin{aligned} \|\mathbf{y} - \mathbf{X}\tilde{\beta}_{\mathbf{S}}\|_2^2 &\leq (1 + \epsilon) \|\mathbf{y} - \mathbf{X}\beta_{lsr}\|_2^2, \\ \|\beta_{lsr} - \tilde{\beta}_{\mathbf{S}}\|_2^2 &\leq \frac{\epsilon}{\sigma_{\min}^2(\mathbf{X})} \|\mathbf{y} - \mathbf{X}\beta_{lsr}\|_2^2 \leq \epsilon \kappa^2(\mathbf{X}) (\gamma^{-2} - 1) \|\beta_{lsr}\|_2^2, \end{aligned}$$

where  $\gamma$  is defined by  $\gamma \leq \|\mathbf{U}_{\mathbf{X}} \mathbf{U}_{\mathbf{X}}^T \mathbf{y}\|_2 / \|\mathbf{y}\|_2 \leq 1$ .

### 3 Preliminaries and Previous Work

For a matrix  $\mathbf{X} = [x_{ij}] \in \mathbb{R}^{n \times d}$ , we let  $\mathbf{x}^{(i)}$  be its  $i$ -th row,  $\mathbf{x}_j$  be its  $j$ -th column,  $\|\mathbf{X}\|_F = (\sum_{i,j} x_{ij}^2)^{1/2}$  be its Frobenius norm, and  $\|\mathbf{X}\|_2 = \max_{\|\mathbf{z}\|_2=1} \|\mathbf{X}\mathbf{z}\|_2$  be its spectral norm. We let  $\mathbf{I}_n$  be an  $n \times n$  identity matrix and let  $\mathbf{0}$  be an all-zero matrix with proper size.

We let the thin singular value decomposition of  $\mathbf{X} \in \mathbb{R}^{n \times d}$  be

$$\mathbf{X} = \mathbf{U}_{\mathbf{X}} \boldsymbol{\Sigma}_{\mathbf{X}} \mathbf{V}_{\mathbf{X}}^T = \sum_{i=1}^d \sigma_i(\mathbf{X}) \mathbf{u}_{\mathbf{X},i} \mathbf{v}_{\mathbf{X},i}^T.$$

Here  $\mathbf{U}_{\mathbf{X}}$ ,  $\boldsymbol{\Sigma}_{\mathbf{X}}$ , and  $\mathbf{V}_{\mathbf{X}}$  are of sizes  $n \times d$ ,  $d \times d$ , and  $d \times d$ , and the singular values  $\sigma_1(\mathbf{X}), \dots, \sigma_d(\mathbf{X})$  are in non-increasing order. We let  $\mathbf{U}_{\mathbf{X}}^\perp$  be an  $n \times (n-d)$  column orthogonal matrix such that  $\mathbf{U}_{\mathbf{X}}^T \mathbf{U}_{\mathbf{X}}^\perp = \mathbf{0}$ . The condition number of  $\mathbf{X}$  is defined by  $\kappa(\mathbf{X}) = \sigma_{\max}(\mathbf{X})/\sigma_{\min}(\mathbf{X})$ .

Based on SVD, the (row) *statistical leverage scores* of  $\mathbf{X} \in \mathbb{R}^{n \times d}$  is defined by

$$l_i = \|\mathbf{u}_{\mathbf{X}}^{(i)}\|_2^2, \quad i = 1, \dots, n.$$

It is obvious that  $\sum_{i=1}^n l_i = d$ . Exactly computing the  $n$  leverages scores costs  $\mathcal{O}(nd^2)$  time, which is as expensive as exactly solving the LSR problem (1).

Fortunately, if  $\mathbf{X}$  is a skinny matrix, the leverages scores can be highly efficiently computed within arbitrary accuracy using the techniques of [2, 3].

There are many ways to construct the random sampling/projection matrix  $\mathbf{S}$ , and below we describe some of them.

- **Uniform Sampling.** The sampling matrix  $\mathbf{S}$  is constructed by sampling  $c$  rows of the identity matrix  $\mathbf{I}_n$  uniformly at random. This method is the simplest and fastest, but in the worst case its performance is very bad [9].
- **Leverage Scores Based Sampling.** The sampling matrix  $\mathbf{S}$  is computed by Algorithm 1;  $\mathbf{S}$  has  $c$  rows in expectation. This method is proposed in [4, 5].
- **Subsampled Randomized Hadamard Transform (SRHT).** The random projection matrix  $\mathbf{S} = \sqrt{n/c}\mathbf{R}\mathbf{H}\mathbf{D}$  is called SRHT [1, 6, 12] if
  - $\mathbf{R} \in \mathbb{R}^{c \times n}$  is a subset of  $c$  rows from the  $n \times n$  identity matrix, where the rows are chosen uniformly at random and without replacement;
  - $\mathbf{H} \in \mathbb{R}^{n \times n}$  is a normalized Walsh–Hadamard matrix;
  - $\mathbf{D}$  is an  $n \times n$  random diagonal matrix with each diagonal entry independently chosen to be  $+1$  or  $-1$  with equal probability.
 SRHT is a fast version of the Johnson-Lindenstrauss transform.
- **Sparse Embedding Matrices.** The sparse embedding matrix  $\mathbf{S} = \Phi\mathbf{D}$  enables random projection performed in time only linear in the number of nonzero entries of  $\mathbf{X}$  [2]. The random linear map  $\mathbf{S} = \Phi\mathbf{D}$  is defined by
  - $h : [n] \mapsto [c]$  is a random map so that for each  $i \in [n]$ ,  $h(i) = t$  for  $t \in [c]$  with probability  $1/c$ ;
  - $\Phi \in \{0, 1\}^{c \times n}$  is a  $c \times n$  binary matrix with  $\Phi_{h(i), i} = 1$ , and all remaining entries 0;
  - $\mathbf{D}$  is the same to the matrix  $\mathbf{D}$  of SRHT.

## 4 Proof

In Section 4.1 we list some of the previous work that will be used in our proof. In Section 4.2 we prove Theorem 1. We prove the theorem by using the techniques in the proof of Lemma 1 and 2 of [5] and Lemma 1 and 2 of [6]. For the sake of self-contain, we repeat some of the proof of [5] in the following paragraphs.

### 4.1 Key Lemmas

**Lemma 1 (Deterministic Error Bound, Lemma 1 and 2 of [6]).** *Suppose we are given an overconstrained least squares approximation problem with  $\mathbf{X} \in \mathbb{R}^{n \times d}$  and  $\mathbf{y} \in \mathbb{R}^n$ . We let  $\beta_{lsr}$  be defined in (1) and  $\tilde{\beta}_{\mathbf{S}}$  be defined in (2), and define  $\mathbf{z}_{\mathbf{S}} \in \mathbb{R}^d$  such that  $\mathbf{U}_{\mathbf{X}}\mathbf{z}_{\mathbf{S}} = \mathbf{X}(\beta_{lsr} - \tilde{\beta}_{\mathbf{S}})$ . Then the following equality and inequalities hold deterministically:*

$$\begin{aligned} \|\mathbf{y} - \mathbf{X}\tilde{\beta}_{\mathbf{S}}\|_2^2 &= \|\mathbf{y} - \mathbf{X}\beta_{lsr}\|_2^2 + \|\mathbf{U}_{\mathbf{X}}\mathbf{z}_{\mathbf{S}}\|_2^2, \\ \|\beta_{lsr} - \tilde{\beta}_{\mathbf{S}}\|_2^2 &\leq \frac{\|\mathbf{U}_{\mathbf{X}}\mathbf{z}_{\mathbf{S}}\|_2^2}{\sigma_{\min}^2(\mathbf{X})}, \end{aligned}$$

$$\|\mathbf{z}_S\|_2 \leq \frac{\|\mathbf{U}_X^T \mathbf{S}^T \mathbf{S} \mathbf{U}_X^\perp \mathbf{U}_X^\perp{}^T \mathbf{y}\|_2}{\sigma_{\min}^2(\mathbf{S} \mathbf{U}_X)}.$$

By further assuming that  $\|\mathbf{U}_X \mathbf{U}_X^T \mathbf{y}\|_2 \geq \gamma \|\mathbf{y}\|_2$ , it follows that

$$\|\mathbf{U}_X^\perp \mathbf{U}_X^\perp{}^T \mathbf{y}\|_2^2 \leq \sigma_{\max}^2(\mathbf{X}) (\gamma^{-2} - 1) \|\boldsymbol{\beta}_{lsr}\|_2^2.$$

*Proof.* The equality and the first two inequalities follow from Lemma 1 of [6]. The last inequality follows from Lemma 2 of [6].

**Lemma 2 (Theorem 7 of [5]).** Suppose  $\mathbf{X} \in \mathbb{R}^{d \times n}$ ,  $\mathbf{Y} \in \mathbb{R}^{n \times p}$ , and  $c \leq n$ , and we let  $\mathbf{S} \in \mathbb{R}^{c \times n}$  be the sampling matrix computed by Algorithm 1 taking  $\mathbf{X}$  and  $c$  as input, then

$$\begin{aligned} \mathbb{E} \|\mathbf{X}^T \mathbf{Y} - \mathbf{X}^T \mathbf{S}^T \mathbf{S} \mathbf{Y}\|_F &\leq \frac{1}{\sqrt{c}} \|\mathbf{X}\|_F \|\mathbf{Y}\|_F, \\ \mathbb{E} \|\mathbf{X}^T \mathbf{X} - \mathbf{X}^T \mathbf{S}^T \mathbf{S} \mathbf{X}\|_F &\leq \mathcal{O}\left(\sqrt{\frac{\log c}{c}}\right) \|\mathbf{X}\|_2 \|\mathbf{X}\|_F. \end{aligned}$$

#### 4.2 Proof of Theorem 1

*Proof.* We first bound the term  $\sigma_{\min}^2$  as follows. Applying a singular value inequality in [8], we have that for all  $i \leq \text{rank}(\mathbf{X})$

$$\begin{aligned} |1 - \sigma_i^2(\mathbf{S} \mathbf{U}_X)| &= |\sigma_i(\mathbf{U}_X^T \mathbf{U}_X) - \sigma_i(\mathbf{U}_X^T \mathbf{S}^T \mathbf{S} \mathbf{U}_X)| \\ &\leq \sigma_{\max}(\mathbf{U}_X^T \mathbf{U}_X - \mathbf{U}_X^T \mathbf{S}^T \mathbf{S} \mathbf{U}_X) \\ &= \|\mathbf{U}_X^T \mathbf{U}_X - \mathbf{U}_X^T \mathbf{S}^T \mathbf{S} \mathbf{U}_X\|_2 \end{aligned}$$

Since the leverage scores of  $\mathbf{X}$  are also the leverage scores of  $\mathbf{U}_X$ , it follows from Lemma 2 that

$$\mathbb{E} \|\mathbf{U}_X^T \mathbf{U}_X - \mathbf{U}_X^T \mathbf{S}^T \mathbf{S} \mathbf{U}_X\|_2 \leq \mathcal{O}\left(\sqrt{\frac{\ln c}{c}}\right) \|\mathbf{U}_X\|_F \|\mathbf{U}_X\|_2 = \mathcal{O}\left(\sqrt{\frac{d \ln c}{c}}\right).$$

It then follows from Markov's inequality that the inequality

$$|1 - \sigma_i^2(\mathbf{S} \mathbf{U}_X)| \leq \delta_1^{-1} \mathcal{O}\left(\sqrt{\frac{d \ln c}{c}}\right)$$

holds with probability at least  $1 - \delta_1$ . When

$$c \geq \mathcal{O}(d \delta_1^{-2} \epsilon_1^{-2} \ln(d \delta_1^{-2} \epsilon_1^{-2})), \quad (4)$$

the inequality

$$\sigma_{\min}^2(\mathbf{S} \mathbf{U}_X) \geq 1 - \epsilon_1 \quad (5)$$

holds with probability at least  $1 - \delta_1$ .

Now we bound the term  $\|\mathbf{U}_{\mathbf{X}}^T \mathbf{S}^T \mathbf{S} \mathbf{U}_{\mathbf{X}}^\perp \mathbf{U}_{\mathbf{X}}^{\perp T} \mathbf{y}\|_2$ . Since  $\mathbf{U}_{\mathbf{X}}^T \mathbf{U}_{\mathbf{X}}^\perp = \mathbf{0}$ , we have that

$$\left\| \mathbf{U}_{\mathbf{X}}^T \mathbf{S}^T \mathbf{S} \mathbf{U}_{\mathbf{X}}^\perp \mathbf{U}_{\mathbf{X}}^{\perp T} \mathbf{y} \right\|_2 = \left\| (\mathbf{U}_{\mathbf{X}}^T) (\mathbf{U}_{\mathbf{X}}^\perp \mathbf{U}_{\mathbf{X}}^{\perp T} \mathbf{y}) - (\mathbf{U}_{\mathbf{X}}^T) \mathbf{S}^T \mathbf{S} (\mathbf{U}_{\mathbf{X}}^\perp \mathbf{U}_{\mathbf{X}}^{\perp T} \mathbf{y}) \right\|_2.$$

Since the leverage scores of  $\mathbf{X}$  are also the leverage scores of  $\mathbf{U}_{\mathbf{X}}$ , it follows from Lemma 2 that

$$\begin{aligned} & \mathbb{E} \left\| (\mathbf{U}_{\mathbf{X}}^T) (\mathbf{U}_{\mathbf{X}}^\perp \mathbf{U}_{\mathbf{X}}^{\perp T} \mathbf{y}) - (\mathbf{U}_{\mathbf{X}}^T) \mathbf{S}^T \mathbf{S} (\mathbf{U}_{\mathbf{X}}^\perp \mathbf{U}_{\mathbf{X}}^{\perp T} \mathbf{y}) \right\|_2 \\ & \leq \frac{1}{\sqrt{c}} \left\| \mathbf{U}_{\mathbf{X}} \right\|_F \left\| \mathbf{U}_{\mathbf{X}}^\perp \mathbf{U}_{\mathbf{X}}^{\perp T} \mathbf{y} \right\|_2 = \sqrt{\frac{d}{c}} \left\| \mathbf{U}_{\mathbf{X}}^\perp \mathbf{U}_{\mathbf{X}}^{\perp T} \mathbf{y} \right\|_2. \end{aligned}$$

It follows from the Markov's inequality that the following inequality holds with probability at least  $1 - \delta_2$ :

$$\left\| \mathbf{U}_{\mathbf{X}}^T \mathbf{S}^T \mathbf{S} \mathbf{U}_{\mathbf{X}}^\perp \mathbf{U}_{\mathbf{X}}^{\perp T} \mathbf{y} \right\|_2 \leq \frac{\delta_2^{-1} \sqrt{d}}{\sqrt{c}} \left\| \mathbf{U}_{\mathbf{X}}^\perp \mathbf{U}_{\mathbf{X}}^{\perp T} \mathbf{y} \right\|_2. \quad (6)$$

Thus when

$$c \geq d \delta_2^{-2} \epsilon_2^{-2} (1 - \epsilon_1)^{-2}, \quad (7)$$

it follows from (5), (6), and the union bound that the inequality

$$\frac{\left\| \mathbf{U}_{\mathbf{X}}^T \mathbf{S}^T \mathbf{S} \mathbf{U}_{\mathbf{X}}^\perp \mathbf{U}_{\mathbf{X}}^{\perp T} \mathbf{y} \right\|_2}{\sigma_{\min}^2(\mathbf{S} \mathbf{U}_{\mathbf{X}})} \leq \epsilon_2 \left\| \mathbf{U}_{\mathbf{X}}^\perp \mathbf{U}_{\mathbf{X}}^{\perp T} \mathbf{y} \right\|_2 \quad (8)$$

holds with probability at least  $1 - \delta_1 - \delta_2$ . We let  $\epsilon_1 = 0.5$ ,  $\epsilon_2 = \sqrt{\epsilon}$ ,  $\delta_1 = \delta_2 = 0.1$ , and let  $\mathbf{z}_{\mathbf{S}}$  be defined in Lemma 1. When

$$c \geq \max \{ \mathcal{O}(d \ln d), 400d\epsilon^{-1} \},$$

it follows from (4), (7), (8), and Lemma 1 that with probability at least 0.8 the following inequality holds:

$$\|\mathbf{z}_{\mathbf{S}}\|_2 \leq \sqrt{\epsilon} \left\| \mathbf{U}_{\mathbf{X}}^\perp \mathbf{U}_{\mathbf{X}}^{\perp T} \mathbf{y} \right\|_2.$$

Since  $\mathbf{U}_{\mathbf{X}}^\perp \mathbf{U}_{\mathbf{X}}^{\perp T} \mathbf{y} = \mathbf{y} - \mathbf{X} \boldsymbol{\beta}_{\text{LSR}}$ , the theorem follows directly from Lemma 1.

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